Topology Preliminary Examination SS2019
University of Cincinnati Department of Mathematical Sciences

1. Which pairs of topological spaces are homeomorphic? Justify.

   (a) $\mathbb{R}$ and $\mathbb{R}^3$

   (b) The plane and the open unit disc centered at $(1, 1)$

   (c) The plane and the closed unit disc centered at $(1, 1)$

   (d) $\mathbb{R} \setminus \{0, 1\}$ and $\mathbb{R} \setminus \{0\}$

   (e) $\mathbb{R}^2 \setminus \{(0, 0), (1, 1)\}$ and $\mathbb{R} \setminus \{(0, 0)\}$

   (f) $\prod_{\mathbb{N}} [-n, n] \subset \prod_{\mathbb{N}} \mathbb{R}$, both with the product topology.

2. Let $p : X \to Y$ be a closed map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. Show that if Y is compact, then X is compact.

3. (a) What does it mean to say that a space X is connected?

   (b) What does it mean to say that a space is path connected?

   (c) Give an example of a connected, but not path connected space.

   (d) Prove: If $U \subset R^n$ is open and connected, then U is path connected.

4. Recall that a space is *Lindelöf* if every open cover contains a countable subcover. Show that every regular *Lindelöf* space is normal.

   Recall a topological space X is regular means: Points are closed and, given a closed set and a point not in it, they can be separated by open sets.

5. $A \subset X$ is a retract if there exists a continuous $f : X \to A$ (called a retraction) so that for each $a \in A$, $f(a) = a$.

   (a) If $A \subset X$ is a retract and $a^* \in A$, show that the homomorphism

   $$h : \pi_1(X, a^*) \to \pi_1(A, a^*)$$

   induced by the retraction is onto.

   (b) Show that $S^1 = \{x \in R^2 : |x| = 1\}$ is not a retract of $D^2 = \{x \in R^2 : |x| \leq 1\}$.